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LETTER TO THE EDITOR

Height correlations in the Abelian sandpile model

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Abstract. We study the distribution of heights in the self-organized critical state of the Abelian sandpile model on a d-dimensional hypercubic lattice. We calculate analytically the concentration of sites having minimum allowed value in the critical state. We also calculate, in the critical state, the probability that the heights, at two sites separated by a distance r, would both have minimum values and show that the lowest-order r-dependent term in it varies as r^{-2d} for large r.

In recent years, the concept of self-organized criticality (SOC) proposed by Bak *et al* [1, 2] has attracted a lot of attention as a possible general framework for explanation of the occurrence of robust power laws in nature as it does not require fine tuning of any parameter to set criticality. Of the several models showing SOC, the sandpile model is simplest in structure and has been studied intensively ([3–9], and references therein).

In an earlier paper, [10] we have shown that the sandpile automaton model has an Abelian group structure which allows a simple characterization of its critical state. Most of the sandpile configurations are forbidden and do not appear in the critical state. All the allowed configurations that do appear in the SOC state do so with equal probability. There is a recursive algorithm [10] (which we call the burning algorithm) to determine whether a given configuration is allowed or forbidden in the SOC state. The total number of allowed configurations in the SOC state can also be computed exactly, and increases as μ^N where N is the number of sites in the lattice and μ is a lattice-dependent constant. Also the correlation function, measuring the expected number of topplings at site j due to a particle added at i, is known in the critical state [10].

It is, however, desirable to have a more direct physical characterization of the SOC state. One of the quantities that may be used to characterize the SOC state is the relative abundances of different heights in the SOC state. For a square lattice, these have been determined numerically by Manna [11] and Erzan and Sinha [12]. For a model with continuous heights these were studied earlier, also numerically, by Zhang [13]. Furthermore, there are correlations between heights of the pile at nearby sites in the critical state. For example, two adjacent sites cannot both have the minimum allowed height in the critical state. So far, these quantities have been calculated analytically only for the Bethe lattice [9].

In this letter, we present some results for the SOC state of the Abelian sandpile model (ASM) on the usual d-dimensional hypercubic lattice. These results are not quite as complete as for the Bethe lattice. In particular we have been able to give analytical formula only for the probability of height at a given site taking its minimum allowed value (1 in our definition), but not for other heights. We show that on a large square lattice, the fractional number of sites having height 1 is $P(1) = (2/\pi^2) - (4/\pi^3) \approx 0.0736$

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and that the joint probability that two sites separated by distance r have heights 1 each is $P_{11}(r) = P^2(1)(1 - 1/2r^4 + \text{higher order terms})$. The treatment can easily be extended to other lattices and higher dimensions. In a d-dimensional hypercubic lattice ($d \geqslant 2$), the connected part of the height-height correlation function varies as r^{-2d}. Our technique can also be used to calculate the probabilities of some other subconfigurations such as two adjacent sites having heights 1 and 2 respectively.

Consider, for definiteness, the ASM on a finite square lattice of size $L \times L$. The height z_i at any site i takes values 1, 2, 3 or 4 in a stable configuration. Particles are added at randomly chosen sites and the addition of a particle increases the height at that site by 1. If this height exceeds the critical value 4, then the site topples, and on toppling its height decreases by 4 and the heights at each of its neighbours increases by 1. The total number of particles (sandgrains) in the system is conserved on topplings at all sites other than the boundary sites.

The burning algorithm to determine if a given configuration of heights occurs with non-zero probability in the SOC state is defined as follows. We simply delete (burn) from a given configuration any site j whose height is strictly greater than the number of its unburnt neighbours (here by neighbours we mean sites connected to j by a bond). The process is repeated until no more sites can be burnt. If and only if this results in eventually all the sites of the lattice being burnt away, the configuration is allowed. If the burning procedure stops with a finite subset of sites remaining unburnt, the configuration is forbidden in the SOC state.

We consider a large $L \times L$ square lattice. The number of allowed configurations N_{total} on this lattice is given by

$$N_{\text{total}} = \det \Delta \tag{1}$$

where Δ is the matrix specifying the toppling rules [10]. In our problem, Δ is an $(L^2 \times L^2)$ discrete Laplacian matrix with

$$\Delta(\boldsymbol{r}, \boldsymbol{r}) = \begin{cases} 4 & \text{if } \boldsymbol{r} = \boldsymbol{r} \\ -1 & \text{if } |\boldsymbol{r} - \boldsymbol{r}| = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Let us consider an allowed configuration C on the square lattice L where the site O has height $z_0 = 1$. Since two adjacent sites cannot both have height 1 in an allowed configuration, all the four neighbours N, E, S and W (see figure 1) of O can only have

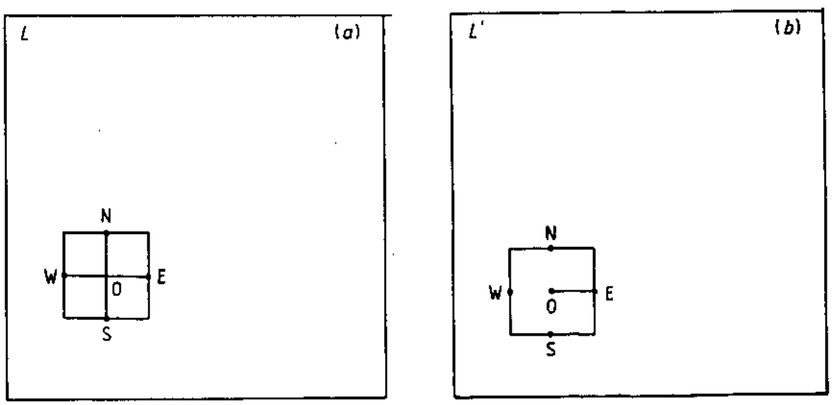

Figure 1. An arbitrary site O and its four neighbours N, W, S and E on the square lattice L. (b) The graph of the lattice L' obtained from L by deleting the bonds ON, OW and OS.

heights 2, 3 or 4 in C. Then consider the configuration obtained by reducing the heights at N, S and W each by 1. Call this configuration C'. Now we consider an ASM defined on a lattice L' obtained from L by deleting three bonds ON, OS and OW so that in this new ASM, toppling at O does not add a particle at N, S or W. For each bond deleted, we also decrease the maximum height allowed at the two end sites of the bond by 1 (so that sand is still conserved at these end points in the new ASM). For this ASM, the toppling rule matrix Δ' is also $L^2 \times L^2$ and is given by

$$\Delta'(\boldsymbol{r}, \boldsymbol{r}) = \Delta(\boldsymbol{r}, \boldsymbol{r}) + B(\boldsymbol{r}, \boldsymbol{r}) \tag{3}$$

with $B(\boldsymbol{r}, \boldsymbol{r}) = 0$ except for the following elements

$$\begin{aligned} &B(\mathrm{N}, \mathrm{N}) = B(\mathrm{W}, \mathrm{W}) = B(\mathrm{S}, \mathrm{S}) = -1 \\ &B(\mathrm{O}, \mathrm{O}) = -3 \\ &B(\mathrm{O}, \mathrm{N}) = B(\mathrm{N}, \mathrm{O}) = B(\mathrm{S}, \mathrm{O}) = B(\mathrm{O}, \mathrm{S}) = B(\mathrm{W}, \mathrm{O}) = B(\mathrm{O}, \mathrm{W}) = 1. \end{aligned} \tag{4}$$

Since C is an allowed configuration on L, all its sites can be burnt under Δ in a burning procedure. Let the sequence in which the burning proceeds be $i_1, i_2, i_3, \ldots$. Then it is easy to see that sites in C' can be burnt under Δ' using the same sequence $i_1, i_2, i_3, \ldots$. Hence C' is an allowed configuration under Δ' iff C is an allowed configuration under Δ. Thus the number of configurations allowed under Δ with height $z_0 = 1$ is equal to the number of all allowed configurations under the toppling rules given by Δ'. The latter, however, is equal to $\det \Delta'$. Hence we get

$$P(1) = \frac{\det \Delta'}{\det \Delta} = \det[I + GB] \tag{5}$$

where the matrix $G = \Delta^{-1}$. The non-zero elements of B occur only in four rows and columns of B and thus one needs to calculate a 4×4 determinant, whose elements are given in terms of the matrix elements of G. The matrix G, by definition, satisfies the two-dimensional discrete Poisson equation

$$\sum_{\boldsymbol{r}'} G(\boldsymbol{r}, \boldsymbol{r}) \Delta(\boldsymbol{r}, \boldsymbol{r}') = \delta_{\boldsymbol{r}, \boldsymbol{r}'}. \tag{6}$$

For large L and with both $\boldsymbol{r}$ and $\boldsymbol{r}'$ away from the boundary, this implies

$$G(\boldsymbol{r}, \boldsymbol{r}') = G(\boldsymbol{r}, \boldsymbol{r}') + \frac{1}{(4\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\cos((x - x')\theta_1 + (y - y')\theta_2) - 1}{1 - \frac{1}{2}(\cos\theta_1 + \cos\theta_2)} \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2. \tag{7}$$

For $\boldsymbol{r}'$ deep inside the lattice, $G(\boldsymbol{r}, \boldsymbol{r}')$ is independent of $\boldsymbol{r}'$ due to translational invariance of the lattice. However it depends on the value of L and diverges as $(\ln L)/2\pi$ for large L. The integrals $\phi(\boldsymbol{r} - \boldsymbol{r}') = G(\boldsymbol{r}, \boldsymbol{r}') - G(\boldsymbol{r}', \boldsymbol{r}')$ can be easily evaluated [14]. We quote here a few values

$$\begin{aligned} &\phi(\pm \boldsymbol{e}_1) = \phi(\pm \boldsymbol{e}_2) = -\frac{1}{4} \\ &\phi(\pm \boldsymbol{e}_1 \pm \boldsymbol{e}_2) = -\frac{1}{\pi} \end{aligned} \tag{8}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit vectors in the x and y directions respectively. Using the values of $\phi(\boldsymbol{r})$ in (5), a straightforward calculation of the 4×4 determinant gives

$$P(1) = \frac{2}{\pi^2}\left(1 - \frac{2}{\pi}\right) \approx 0.073\,636. \tag{9}$$

This is in good agreement with the numerical value $P(1) = 0.0736 \pm 0.003$ found in [11]. Using the same technique, $P(1)$ can be evaluated in higher dimensions also.

In a similar way, we can calculate the probability of occurrence of subconfigurations of the type shown in figure 2. They have the property that decreasing the height at any site by 1 makes them forbidden. Consider a particular subconfiguration S of this type with a given set of heights (for example, $1-2$). Let $\{C\}$ be the set of all allowed configurations on the lattice L in which S occurs. Now we construct a graph L' in the following way. We start trying to delete the boundary bonds connecting the sites in S to the rest of the lattice L one by one. Deletion of each such bond is accompanied by the reduction of maximum height by 1 at both ends of the bond and the deletion is allowed iff the coordination number of any site in S does not become less than its specified height. This process stops when the coordination number of each boundary site in S becomes equal to its height specified in C on L. This results in only one of the bonds connecting the sites in S to the rest of the lattice remaining undeleted. Arguing as before, the set of all allowed configurations on the graph L' are in one-to-one correspondence with the set $\{C\}$. Thus, in general, the probability of occurrence of any such subconfiguration S is given by

$$\mathrm{Prob}(S) = \det[I + GB] \tag{10}$$

where the matrix B is S dependent and has only a finite number of non-zero rows and columns.

For the subconfiguration S_1 shown in figure 2(a), B_1 is a 7×7 matrix. Then an explicit evaluation of the 7×7 determinant in (10), using Mathematica [15] and the values of G from (7), gives

$$\mathrm{Prob}(S_1) = \frac{9}{32} - \frac{9}{2\pi} + \frac{47}{2\pi^2} - \frac{48}{\pi^3} + \frac{32}{\pi^4} \approx 0.010\,3411. \tag{11}$$

A similar calculation for the subconfiguration S_2 (figure 2(b)) involves a 10×10 matrix B_2 and we get

$$\mathrm{Prob}(S_2) = -\frac{81}{14} + \frac{525}{4\pi} - \frac{1315}{\pi^2} + \frac{60\,076}{9\pi^3} - \frac{503\,104}{27\pi^4} + \frac{257\,024}{9\pi^5} - \frac{1785\,856}{81\pi^6} + \frac{524\,288}{81\pi^7} \approx 0.001\,419\,94. \tag{12}$$

For the subconfiguration S_3 (figure 2(c)), the matrix B_3 is nine dimensional. Explicit

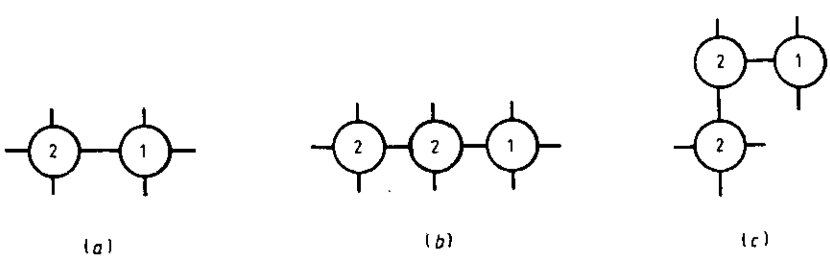

Figure 2. Examples of subconfigurations of the type that become forbidden on decreasing the height by 1 at any site.

computation of the determinant gives

$$\mathrm{Prob}(S_3) = -\frac{23}{16} + \frac{389}{12\pi} - \frac{2576}{9\pi^2} + \frac{1280}{\pi^3} - \frac{3072}{\pi^4} + \frac{11\,264}{3\pi^5} - \frac{16\,384}{9\pi^6}$$
$$\approx 0.001\,344\,77. \tag{13}$$

For larger subconfigurations of this type, the evaluation of the determinants becomes very tedious.

Since all allowed configurations with $z_0 < 4$ remain allowed if z_0 is increased, it is easy to see that

$$P(1) \leq P(2) \leq P(3) \leq P(4).$$

A typical allowed configuration with $z_0 = 2$ could be either an allowed configuration even with $z_0 = 1$ or becomes disallowed with $z_0 = 1$. In the latter case, it is a configuration of the type shown in figure 2. Hence

$$P(2) = P(1) + 4\,\mathrm{Prob}(C_1) + 4\,\mathrm{Prob}(C_2) + 8\,\mathrm{Prob}(C_3) + \ldots. \tag{14}$$

This is a series of positive terms and using the already calculated values of the first four terms we get

$$P(2) \geq 0.131\,438. \tag{15}$$

This should be compared to the numerical value $P(2) \approx 0.174$ obtained in [11]. The series in (14) seems to converge very slowly.

We now calculate $P_{11}(r)$, i.e. the probability that two sites O and O′ (both deep inside the lattice) separated by distance r will both have height 1 in the SOC state. The neighbours (figure 3(a)) of O are N, E, S, W and those of O′ are N′, E′, S′ and W′ respectively. The allowed configurations on lattice L with heights 1 at O and O′ are in one-to-one correspondence with the allowed configurations on a lattice L' with the bonds ON, OW, OS, O′N′, O′W′ and O′S′ deleted (figure 3(b)) and the maximum allowed heights at both end points of the deleted bonds being reduced by 1. Then

$$P_{11}(r) = \det[I + GB_{11}] \tag{16}$$

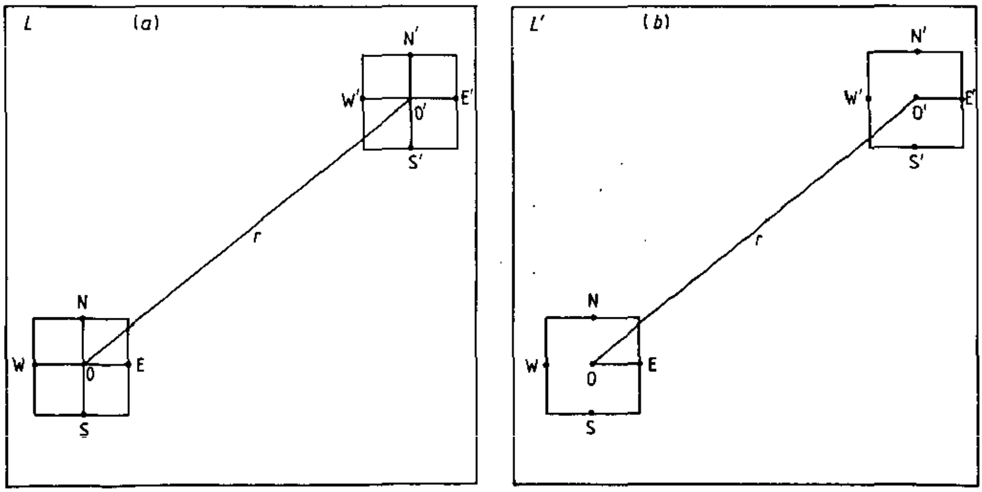

Figure 3. (a) Figure showing sites O and O′ and their respective neighbours on the square lattice L. The distance between O and O′ is r. (b) The lattice L' obtained by deleting the bonds ON, OW, OS, O′N′, O′W′ and O′S′ from figure 3(a).

where B_{11}, in this case, is an 8×8 matrix. This determinant simplifies to the determinant of a 6×6 matrix M of the form

$$M = \begin{pmatrix} P & Q \\ Q' & P \end{pmatrix} \tag{17}$$

where P and Q are 3×3 matrices and Q' is the transpose of Q. Elements of P are independent of r and elements of Q tend to zero as r^{-2} for large r. Keeping only these leading terms, we get

$$P_{11}(r) = P^2(1)\left(1 - \frac{1}{2r^4} + \text{higher order terms}\right) \tag{18}$$

where $P(1)$ is given by (9).

In higher dimensions also, one can express $\det[I + GB]$ in terms of the matrices P and Q. In d dimensions, elements of Q vary as r^{-d} for large r. The lowest-order r-dependent term in $P_{11}(r)$ varies as $Q^2(r)$ and hence as r^{-2d}. In the case of the Bethe lattice, the effective dimension is infinite and the anticorrelation decays exponentially, in agreement with our earlier results [9].

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